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STRICTLY ERGODIC MODELS FOR NON-INVERTIBLE TRANSFORMATIONS

BY

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ABSTRACT

We generalize a result of R. Jewett [J]: If T is an ergodic measure preserving transformation on (X, Ω, λ), T not necessarily invertible, there exists a strictly ergodic S acting on (Y, Θ, ν), where Y is compact, such that (X, Ω, λ, T) is measure theoretically isomorphic to (Y, Θ, ν, S).

I. Introduction

A continuous transformation of a compact metric space Y is said to be strictly ergodic, if there is a unique Borel probability measure ν, fixed by the action, and $\nu(U) > 0$ for every non-empty open set $U \subset Y$. In 1969, R. Jewett [J] proved that every weakly mixing invertible transformation on a Lebesgue space is measure isomorphic to a strictly ergodic transformation. Krieger [K] proved it for every ergodic invertible transformation in 1970. Several people obtained similar results with different proofs but always for invertible transformation ([H–R], [B–F]). Finally, in 1983, B. Weiss [W] extended this result to any ergodic, free action of an elementary amenable group. Using Weiss's proof, I extended this for every ergodic, free action of a general discrete amenable group [Rs1]. Using Weiss's method, I was able to obtain a similar result in the case of a non-invertible transformation and this is the purpose of the following work.

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II. Construction of uniform, separated towers

Let (X, Ω, λ) be a Lebesgue space. In the sequel, T will be an ergodic measure preserving transformation on (X, Ω, λ), that is $\lambda(A) = \lambda(T^{-1}A)$ for any A in Ω. Recall first Rohlin's lemma in the case of an ergodic transformation that may be non-invertible.

LEMMA 1. *For every n and every δ there exists a set B in Ω such that*:
(i) *The sets $T^{-1}B$, $0 \leqq i \leqq n-1$ are all disjoint.*
(ii) $\lambda(\bigcup_{i=0}^{i=n-1} T^{-i}B) \geqq 1-\delta$.

Note in this case that the tower is to be considered from top to bottom (one takes successive inverse images of the set B). B will be called the ceiling of the tower, $T^{-(n-1)}B$ its base. The proof of Lemma 1 is identical to the invertible case. Just to recall it, take some set A of measure smaller than δ/n, and look at

$$A_1 = T^{-1}A \setminus A, \dots, A_k = T^{-1}A_{k-1} \setminus A \cdots.$$

If $B = A_n \cup A_{2n} \cup A_{3n} \cdots \cup A_{kn} \cdots$, it is easy to see that B satisfies the conclusion of the lemma.

Another difference to note between this and the invertible case is that in the tower (and more generally in X), a point can have several inverse images. The proof will now very much parallel Weiss's proof in the invertible case [W].

LEMMA 2. *Given $M > 0$, there exists an increasing sequence of integers $N_1, N_2, \dots, N_i, \dots$ and there exists a sequence of sets B_i, ceilings of Rohlin towers $F_i = \bigcup_{l=0}^{l=N_i} T^{-l}B_i$ such that*:
(a) $\lambda(F_i) \to 1$ *as* $i \to +\infty$.
(b) *For any* $i < k$, $F_i \subset \bigcup_{l=M}^{l=N_k-M} T^{-l}B_k$.
One will say that the towers are M-separated.

PROOF. Let $N_1 > M$ be fixed. Let $F_1^{(1)} = \bigcup_{l=0}^{l=N_1} T^{-l}B_1^{(1)}$ be an ordinary Rohlin tower with $\lambda(F_1^{(1)}) \geqq 1-\frac{1}{3}$. The towers will be built by an inductive procedure. Let $F_2^{(2)} = \bigcup_{l=0}^{l=N_2} T^{-l}B_2^{(2)}$ be such that $\lambda(F_2^{(2)}) > 1 - 1/(2 \times 3^2)$ where N_2 is chosen so that

$$(N_1 + 2M) \leqq N_2/100.$$

Let us say that condition (c) holds at step n if, for any $i < k \leqq n$,

$$F_i^{(n)} \subset \bigcup_{l=M}^{l=N_k-M} T^{-l}B_k^{(n)}.$$

Let $B_1^{(2)}$ be

$$B_1^{(1)} \cap \bigcup_{l=M}^{l=N_2-N_1-M} T^{-l}B_2^{(2)}.$$

This defines a tower $F_1^{(2)}$. The way $B_1^{(2)}$ was defined, it is clear that condition (c) holds at this step. Because N_1/N_2 was chosen small enough, $\lambda(F_1^{(2)}) > 1 - \frac{1}{3} - \frac{1}{3^2}$.

($F_1^{(1)} - F_1^{(2)}$ lies in $X - F_2^{(2)}$ or in

$$\bigcup_{l=0}^{l=M} T^{-l}B_2^{(2)} \quad \bigcup_{l=N_2-N_1-M}^{l=N_2} T^{-l}B_2^{(2)}.\Bigg)$$

By induction at step n, let the towers

$$F_j^{(n-1)} = \bigcup_{l=0}^{l=N_j} T^{-l}B_j^{(n-1)} \qquad \text{for } j \leqq n-1$$

satisfy condition (c) and

$$\lambda(F_j^{(n-1)}) > 1 - \sum_{l=j}^{l=n-1} 1/3^l \qquad (\text{for any } j \leqq n-1),$$

$$\lambda(F_{n-1}^{(n-1)}) > 1 - 1/(2 \times 3^{n-1}).$$

Let N_n be fixed much bigger than N_{n-1}. Let $F_n^{(n)}$ be a Rohlin tower

$$F_n^{(n)} = \bigcup_{l=0}^{l=N_n} T^{-l}B_n^{(n)}$$

such that $\lambda(F_n^{(n)}) > 1 - 1/(2 \times 3^n)$. One defines $B_j^{(n)}$ for $j \leqq n-1$ by

$$B_j^{(n)} = B_j^{(n-1)} \cap \bigcup_{l=M+\Sigma_{k=j+1}^{k=n-1} N_k}^{l=N_n-N_{n-1}-M} T^{-l}B_j^{(n-1)} \qquad \text{for } j < n-1$$

and

$$B_{n-1}^{(n)} = B_{n-1}^{(n-1)} \cap \bigcup_{l=M}^{l=N_n-N_{n-1}-M} T^{-l}B_n^{(n)}.$$

One checks that condition (c) is satisfied at step n as follows: Let $j < k$ be fixed. If $k = n$, by definition of $B_j^{(n)}$ it is an easy matter to check

$$F_j^{(n)} \subset \bigcup_{l=M}^{l=N_n-M} T^{-l}B_k^{(n)}.$$

If $k < n$, this condition was true at step $n-1$, so the only possibility for it to be false is that one "erased" some tower $F_k^{(n-1)}$ but left some tower $F_j^{(n)}$ in it. This can only happen in the first $M + \Sigma_{k'=k+1}^{k'=n-1} N_{k'} + N_k$ levels of $F_n^{(n)}$ or in the last $N_{n-1} + M$ of it. The definition of $B_j^{(n)}$ implies now the result. If N_n was chosen big enough one obtains

$$\lambda\left(\bigcup_{l=0}^{l=N_j} T^{-l}B_j^{(n)}\right) > 1 - \sum_{m=j}^{m=n} 1/3^m.$$

The inductive procedure can thus be carried on. For any j, one can define $B_j = \lim_n B_j^{(n)}$. B_j is the ceiling of a tower F_j of height N_j such that

$$\lambda(F_j) > 1 - \sum_{m=j}^{m=+\infty} 1/3^m$$

so that (a) is true. Because $B_j = \bigcap_{n \geqq j} B_j^{(n)}$ it is easy to check that condition (b) is true (condition (c) holds at any step).

Definition. An (n, M, δ) uniform Rohlin tower F is a tower of height n, $F = \bigcup_{j=0}^{j=n-1} T^{-j}B$ that satisfies $(1/M)\,\Sigma_{j=0}^{j=M-1} 1_F(T^jx) > 1 - \delta$ for almost every x in X.

Theorem 1. *For any n and δ, if M is big enough ($2n/M \leqq \delta$) there exists an (n, M, δ) uniform tower.*

Proof. Let n and δ be given. Let M be such that $2n/M \leqq \delta$. By Lemma 2, there is a sequence $(F_i)_{i \text{ in } N}$ of Rohlin towers of height N_i that are $(M+n)$-separated and $\lambda(F_i) \to 1$. We want to define the ceiling B of the uniform tower. We will do it successively, on every tower F_i.

We first put in B

$$E_1 = \bigcup_{j=1}^{j=[N_1/n]-1} T^{-jn}B_1. \tag{1}$$

(The B_i's are the ceilings of the towers F_i.) That is every n-th level of the first tower F_1. We point out the fact that at this stage, B_1 (corresponding to $j=0$ in (1)) is not in B.

In Step 2, we add to B

$$(2) \qquad E_2 = \bigcup_{j=1}^{j=[N_2/n]-1} T^{-jn}B_2 \cap (X - F_1).$$

Because in Step 1 the first n levels of F_1 were left outside E_1, $E_1 \cup E_2$ is the ceiling of a Rohlin tower of height n. For step k we similarly add to B

$$E_k = \bigcup_{j=1}^{j=[N_k/n]-1} T^{-jn}B_k \cap \left(X - \bigcup_{l=1}^{l=k-1} F_l\right).$$

Because, for each $l < k$, the first n levels of F_l were left outside E_l and the fact that the towers F_l are $(M+n)$-separated, it is easy to see that $\bigcup_{l=1}^{l=k} E_l$ is the ceiling of a Rohlin tower of height n. Finally, if $B = \bigcup_l E_l$, B is the ceiling of a Rohlin tower. Because $\lambda(F_i) \to 1$ and $N_i \to +\infty$, for almost every x, there exists i such that $\bigcup_{j=0}^{j=M-1} T^jx \subset F_i$. By our construction, if $F = \bigcup_{j=0}^{j=n-1} T^{-j}B$, if $E(x) = \{\bigcup T^lx\}_{l=0}^{l=M-1}$ and $E(X)$ is not entirely in F, one of the M first direct images of x under T must be on the n-boundary of some tower F_l, for $l < i$ (that is, it is in the first or last n levels of F_l). Because these towers are $(M+n)$-separated, this can only happen for one such l and, for that given l, the worst case is when $E(x)$ intersects two such F_l-towers, because these two F_l-towers are close, $E(x)$ intersecting the first tower in some of its n first levels and intersecting the second tower in some of its n last levels. We then easily obtain

$$(1/M) \sum_{j=0}^{j=M-1} 1_F(T^jx) \geqq (M-2n)/M > 1 - \delta$$

and this proves the theorem.

Definition. One will say that some Rohlin tower G is *M-away* from some set B in Ω, if for any x in G, $\bigcup_{j=-M}^{j=M} \{T^jx\} \cap B = \varnothing$.

In the sequel we will denote the distance between Lebesgue sets by $d(A, B)$, to mean the measure of the set $(A \setminus B) \cup (B \setminus A)$.

Lemma 3. *There exists a sequence $(F_i')_{i \geq 1}$ of $(n_i', M_i, 3\delta_i)$ uniform towers with ceiling B_i' such that:*

For almost every x in X, for every i in N, there exists at most one $k > i$, such *that in* $\{T^jx\}_0^{M_i}$, *one encounters the boundary* $(B'_k \cup T^{-N_k}B'_k)$ *of the tower* F'_k. *If k exists, in* $\{T^jx\}_0^{M_i}$, *one encounters the boundary of* F'_k *at most twice.*

PROOF. In the case of a non-invertible transformation, our proof is not as direct as it is in the invertible case because, for some ceiling C of a Rohlin tower, one does not know if TC is measurable, and even if it is one does not know the measure of TC. At each stage k, we want to obtain the following:

(H_k) a sequence of Rohlin towers $G_j^{(k)}$ with ceiling $C_j^{(k)}$ for $j \leqq k$, such that for any i, j, $i < j$, for any x in $G_i^{(k)}$:

$$\bigcup_{l=-2M_{i-1}}^{l=2M_{i-1}} \{T^lx\} \cap (T^{-n_j}C_j^{(k)} \cup T^{-n_{j-1}-2M_{j-2}}C_j^{(k)}) = \varnothing .$$

This means that $G_i^{(k)}$ towers are at least $2M_{i-1}$-away from $T^{-n_j}C_j^{(k)} \cup T^{-n_{j-1}-2M_{j-2}}C_j^{(k)}$ (one takes M_0 to be 0). Note that for $l < 0$, $\{T^lx\}$ is a set that may have more than one element.

The proof is carried out by an inductive procedure; at the end of it, we will have $C_i^{(\infty)} = \lim_{k\to\infty} C_i^{(k)}$ and will define

$$B'_i = T^{-n_{i-1}-2M_{i-2}}C_i^{(\infty)} \quad \text{and} \quad n'_i = n_i - n_{i-1} - 2M_{i-2}.$$

This, together with (H_k) for all k, will ensure that the conclusions of the lemma are true.

By Theorem 1, there exists a sequence $(G_i)_{i\geqq 1}$ of (n_i, M_i, δ_i) uniform towers with ceiling C_i. We suppose that $n_1 \ll M_1 \ll n_2 \ll M_2 \cdots$, and that δ_i tends to zero. We choose $(\delta_i)_{i\geqq 1}$ so that $\Sigma_{i=1}^{i=+\infty} \delta_i < +\infty$. The assumptions $2n_i/M_i \leqq \delta_i$, $(n_{i-1} + 2M_{i-2})/n_i < \delta_{i-1}/10$ and $4M_{i-1} \leqq M_i$ are made in the sequel.

The construction will be made so that $d(G_j^{(i-1)}, G_j^{(i)}) < \delta_{i-1}$ for any $i, j, j < i$, in which case $C_i^{(\infty)} = \lim_{k\to\infty} C_i^{(k)}$ exists. We want to change this sequence of towers to obtain a property of separation (and keep the uniformity).

Step 1. We first erase, from C_1, $C_1 \cap \bigcup_{j=0}^{j=n_1} T^{-j}C_2$ and $C_1 \cap \bigcup_{j=n_2-n_1}^{j=n_2} T^{-j}C_2$, namely the first and last n_1 levels of G_2. If n_2 is big enough, this is only a small part of C_1 and this defines $C_1^{(2)}$ and $G_1^{(2)} = \bigcup_{j=0}^{j=n_1} T^{-j}C_1^{(2)}$; $G_1^{(2)}$ remains $(n_1, M_1, 2\delta_1)$ uniform because, for any x in the first M_1 images of x under T, we erased at most two "towers like G_1" (we recall $2n_1/M_1 \leqq \delta_1$). This way, for any x in $G_1^{(2)}$, x is not in the first n_1 levels of G_2, and x and its first n_1 images under T are not in $T^{-n_2}C_2$. Such a construction is necessary because T is not invertible and, for instance, TC may not be in Ω. (H_2) is then easily seen to be true (with $G_2^{(2)} = G_2$).

Step 2. Let us define $C_2^{(3)}$. For this, one deletes from C_2 that part of it in $\bigcup_{j=n_3-(n_2+2M_1)}^{j=n_3+2M_1} T^{-j}C_3$ and also that part of it in $\bigcup_{j=0}^{j=n_2+4M_1} T^{-j}C_3$. The above will ensure that the tower $G_2^{(3)}$ so defined is $2M_1$ away from $T^{-n_3}C_3 \cup T^{-n_2-2M_1}C_3$. To define $C_1^{(3)}$, one erases from C_1

$$C_1 \cap \bigcup_{j=0}^{j=n_1} T^{-j}C_2^{(3)} \quad \text{and} \quad C_1 \cap \bigcup_{j=n_2-n_1}^{j=n_2} T^{-j}C_2^{(3)}$$

(to keep the same property as in step 1) and also that part of C_1 in $\bigcup_{j=0}^{j=n_1} T^{-n_3+j}C_3$ and in $\bigcup_{j=n_2+2M_1-n_1}^{j=n_2+2M_1} T^{-j}C_3$. This will ensure that $G_1^{(3)}$ is 0-away (the above definition is also meaningful for $M=0$) from $T^{-n_3}C_3 \cup T^{-n_2-2M_1}C_3$. This defines $C_1^{(3)}$ and $G_1^{(3)}$. The fact that $G_2^{(3)}$ is (n_2, M_2, δ_2) uniform is easily checked. $G_1^{(3)}$ remains (n_1, M_1, δ_1) uniform because, for almost every x in X, if in the first M_1 images of x towers G_1 were erased because of some G_3-tower and because of some $G_2^{(3)}$-tower, then the $G_2^{(3)}$-tower would be closer than $M_1 \leqq 2M_1$ from $T^{-n_3}C_3 \cup T^{-n_2-2M_1}C_3$ and so should have been erased in this step, hence this situation cannot happen. A last thing to note is that $d(G_1^{(3)}, G_1^{(2)}) \leqq \delta_2$, because $(n_2+2M_1)/n_3 < \delta_2/10$ by hypothesis. (H_3) is easily seen to be true.

Step $k-1$. By induction, suppose that (H_{k-1}) is true and

$$d(G_j^{(k-2)}, G_j^{(k-1)}) < \delta_{k-2} \qquad \text{for any } j, \quad k \leqq k-2.$$

Let us consider $G_k = G_k^{(k)}$. To define $C_{k-1}^{(k)}$, one deletes from C_{k-1} that part of it in $\bigcup_{j=0}^{j=n_{k-1}+4M_{k-2}} T^{-j}C_k$ and in $\bigcup_{j=n_k-(n_{k-1}+2M_{k-2})}^{j=n_k+2M_{k-2}} T^{-j}C_k$. This defines also $G_{k-1}^{(k)}$; it will be $2M_{k-2}$ away from $T^{-n_k}C_k \cup T^{-n_{k-1}-2M_{k-2}}C_k$.

Let us now consider C_{k-2} and delete from it that part in $\bigcup_{j=n_{k-1}+2M_{k-2}-(2M_{k-3}+n_{k-2})}^{j=n_{k-1}+2M_{k-2}+2M_{k-3}} T^{-j}C_k$ and in $\bigcup_{j=n_k-(n_{k-2}+2M_{k-3})}^{j=n_k+2M_{k-3}} T^{-j}C_k$. We also delete from C_{k-2} the part of it near the "boundaries" of $G_{k-1}^{(k)}$, that is the part in $\bigcup_{j=0}^{j=n_{k-2}+4M_{k-3}} T^{-j}C_{k-1}^{(k)}$ and in $\bigcup_{j=n_{k-1}-(n_{k-2}+2M_{k-3})}^{j=n_{k-1}+2M_{k-3}} T^{-j}C_{k-1}^{(k)}$. These two deletions (those due to G_k and those due to $G_{k-1}^{(k)}$) define $C_{k-2}^{(k)}$ and $G_{k-2}^{(k)}$.

We go on with the same procedure for $j = k-3, k-4, \ldots, 1$ where, at each stage j, we define $C_j^{(k)}$ by erasing from C_j that part of it near the "boundaries" of some $G_l^{(k)}$ for $l > j$ (the expression "near the boundaries" being understood as above for C_{k-2}).

This constructin was made so that (H_k) is true. It is also easy to see that $d(G_j^{(k-1)}, G_j^{(k)}) < \delta_{k-1}$; this comes from $(n_{k-1}+2M_{k-2})/n_k < \delta_{k-1}/10$. For $j < k$, $G_j^{(k)}$ remains $(n_j, M_j, 2\delta_j)$-uniform because, for almost every x in X in the M_j first direct images of x under T, we at most erased two towers G_j because for some given l, $l > j$, they were "near the boundaries" of some tower $G_l^{(k)}$. In fact, suppose on the contrary that there exist $l < l'$ such that we erased G_j towers because of $G_l^{(k)}$ and of $G_{l'}^{(k)}$, then some y in $G_l^{(k)}$ would be closer than $2M_{j-1} + M_j + 2M_{j-1} \leqq 2M_{l-1}$ from $T^{-n_{l'}}C_{l'}^{(k)} \cup T^{-n_{l'-1}-2M_{l'-2}}C_{l'}^{(k)}$ and this is absurd by construction.

Because $\Sigma_{i=1}^{i=+\infty} \delta_i < +\infty$ and $(n_{i-1} + 2M_{i-2})/n_i < \delta_{i-1}/10$, by the Borel–Cantelli lemma, the process converges:

For almost every x, after a certain step $k(x)$, either x is in $G_l^{(j)}$ or x is not in $G_l^{(j)}$ for $j \geqq k(x)$, $l \leqq j$, one of these two relations being true independently of j.

Letting $C_j^{(\infty)} = \lim_k C_j^{(k)}$, $B_j' = T^{-n_{j-1}-2M_{j-2}}C_j^{(\infty)}$, $n_j' = n_j - n_{j+1} - 2M_{j-2}$ and $F_j' = \bigcup_{l=0}^{l=n_j'} T^{-1}B_j'$, the properties we are looking for are satisfied: Because, for almost every x, the towers to which x belongs do not change after a given step, it is enough to check these properties at a given finite step and this is what was done in the above proof at step $k-1$. Finally, by changing n_j into n_j', from every tower $G_j^{(\infty)}$ we erased a portion smaller than δ_j so that F_j' is still $(n_j', M_j, 3\delta_j)$-uniform.

LEMMA 4. *For any finite partition $P = (p_0, \ldots, p_k)$, any $\alpha > 0$, there exists n and $\delta > 0$ so that, if B is the ceiling of a Rohlin tower of height $n+1$ such that $\lambda(\bigcup_{j=0}^{j=n} T^{-j}B) > 1 - \delta$, there exists a set $C \subset B$, $\lambda(C) > (1-\alpha)\lambda(T^{-n}B)$, and for any x in C*

$$\sum_{i=0}^{i=k} \left| 1/n+1 \sum_{l=0}^{l=n} 1_{p_i}(T^l x) - \lambda(p_i) \right| < \alpha. \tag{3}$$

PROOF. To prove the lemma it is enough to have

$$(1/\lambda(B)) \sum_{i=0}^{i=k} \int_{T^{-n}B} \left| 1/n+1 \sum_{l=0}^{l=n} 1_{p_i}(T^l x) - \lambda(p_i) \right| d\lambda(x) \leqq \alpha^2.$$

By the mean ergodic theorem, for any γ, there exists r so that

$$\sum_{i=0}^{i=k} \int_X \left| 1/r \sum_{l=0}^{l=r-1} 1_{p_i}(T^l x) - \lambda(p_i) \right| d\lambda(x) \leqq \gamma. \tag{4}$$

If n is such that $r \ll n$, one has

$$(1/\lambda(B))\sum_{i=0}^{i=k}\int_{T^{-n}B}\left|(1/(n+1))\sum_{l=0}^{l=n}1_{p_i}(T^l x)-\lambda(p_i)\right|d\lambda(x)$$

$$\leqq \sum_{i=0}^{i=k}(1/(\lambda(B)\cdot(n+1)))\sum_{l=0}^{l=n}\int_{T^{-n}B}\left|(1/r)\sum_{m=0}^{m=r-1}1_{p_i}(T^{l+m}x)-\lambda(p_i)\right|d\lambda(x)$$

$$+O(r/n).$$

But x is in $T^{-n}B$ is equivalent to $T^l x$ in $T^{l-n}B$ for $l \leqq n$. Because T is measure preserving, one thus obtains

$$\sum_{i=0}^{i=k}(1/(\lambda(B)\cdot(n+1)))\sum_{l=0}^{l=n}\int_{T^{l-n}B}\left|(1/r)\sum_{m=0}^{m=r-1}1_{p_i}(T^m y)-\lambda(p_i)\right|d\lambda(y)$$

$$=\sum_{i=0}^{i=k}(1/(\lambda(B)\cdot(n+1)))\int_{\cup_{l=0}^{l=n}T^{-l}B}\left|(1/r)\sum_{m=0}^{m=r-1}1_{p_i}(T^m y)-\lambda(p_i)\right|d\lambda(y).$$

By (4) and the fact that $(n+1)\lambda(B) > 1-\delta$, one obtains the desired conclusion: if γ, then r and n were properly chosen.

In the sequel, we will use results of Rohlin [Ro]. For the sake of completeness we will first recall the following definition from Rohlin [Ro, pp. 5–6] of an irreducible set (in fact we will only use its properties):

Let T be a homomorphism of a Lebesgue space X onto another Lebesgue space X', let ∂ be a decomposition (see [Ro, p. 3] for a precise definition of this term) of the space X into preimages of points with respect to the homomorphism T, and let $\{\lambda_C\}$ be the canonical system of measures (see [Ro, p. 5] for a precise definition of this) belonging to the decomposition ∂. For an arbitrary measurable set $B \subset X$, let Z denote the union of all elements C of the decomposition ∂ that intersect B, let Z_0 denote the union of all elements $C \subset Z$ for which $\lambda_C(B \cap C) = 0$ and let Z_1 denote the union of all elements C for which $\lambda_C(B \cap C) > 0$. It is clear that the sets Z, Z_0, Z_1 are preimages and that $Z_1 + Z_0 = Z$.

Since $\lambda_C(B \cap C)$ is a measurable function in X/∂, the sets Z_1 and TZ_1 are always measurable, while the sets Z_0, Z, TZ are simultaneously either measurable or non-measurable. The set B is said to be irreducible with respect to the homomorphism T if Z_0 is a set of measure zero. Then the following holds:

(a) If the set B in Ω is irreducible with respect to T, its image TB is measurable.

(b) If the set B is irreducible with respect to T, every subset $B' \subset B$, differing from B by a set of measure zero, is also irreducible with respect to T.

(c) Every measurable set B contains a subset differing from B by a set of measure zero and irreducible with respect to T.

(d) Every measurable set B contains a subset B' differing from B by a set of measure zero and, for each n in N, B' is irreducible with respect to T^n (this comes from (b) and (c)).

These results will enable one to do the following: If F is a Rohlin tower with base $T^{-n}B$, for every finite D, for every finite family of subsets $(C_i)_{i \text{ in } D}$, of $T^{-n}B$, one can replace these subsets by irreducible subsets (with respect to all the T^n) $(C_i')_{i \text{ in } D}$, that is $C_i' \subset C_i$, so that one can suppose in the sequel that all the subsets C of $T^{-n}B$ that one will use are measurable together with all their direct images (one will only use a finite number of subsets). Furthermore, because of (b), if one then considers a subset of C differing from C by a set of measure zero, it remains irreducible. More explicitly, if C is a union $C_1 \cup C_2 \cup \cdots \cup C_k$ of measurable sets, one can first replace C by C' irreducible with respect to all the T^n. Then, each $C' \cap C_i$ $(1 \leqq i \leqq k)$ can be replaced by its irreducible parts C_i'. Using the fact that the countable union of sets of measure zero is of measure zero, it is easy to see that $\bigcup C_i'$ will be irreducible. One will not go back to these facts in the sequel, and suppose that each time, when necessary, one takes the irreducible part.

A last point to be noted is that, in a Rohlin tower with base $T^{-n}B$, if $D \subset T^{-n}B$ and differs from $T^{-n}B$ by a set of measure zero and $D = \bigcup_{i=1}^{i=k} C_i'$(if, as above, $T^{-n}B = C_1 \cup C_2 \cup \cdots \cup C_k$ and we replace every C_1 by some irreducible part C_i'), then

$$\lambda(TD) = \lambda(T^{-1}(TD)) \geqq \lambda(D) = \lambda(T^{-n}B) = \lambda(B).$$

This is because T is measure preserving and $D \subset T^{-1}(TD)$. From $TD \subset T^{-(n-1)}B$ one gets $\lambda(TD) \leqq \lambda(B)$ so, finally, $\lambda(TD) = \lambda(D) = \lambda(B)$; hence taking irreducible parts in a Rohlin tower does not change measures.

DEFINITION. A partition P is said to be uniform if, for any n in N, any $\delta > 0$, there exists k in N such that, for any atom a of $\bigvee_{j=0}^{j=n-1} T^{-j}P$, for almost every x in X, the following holds:

$$\left| (1/k) \sum_{l=0}^{l=k-1} 1_a(T^l x) - \lambda(a) \right| \leqq \delta. \tag{5}$$

For n and k given, P is said to be (n, k, δ)-uniform if (5) is true for every atom of $\bigvee_{j=0}^{j=n-1} T^{-j}P$.

THEOREM 2. *There exists a sequence* $(Q_i)_{i \text{ in } N}$ *of partitions such that*:
(a) *For every* i, Q_i *is uniform*.
(b) $\bigvee_{i=1}^{\infty} Q_i = \Omega$.
(c) *For every* i, $Q_i \subset Q_{i+1}$.

PROOF. The partitions Q_i will be built by an inductive procedure. At step k, one will have partitions $Q_i^{(k)}$ for $i \leqq k$ and α_i, $\Sigma_{i=1}^{\infty} \alpha_i < +\infty$ and they will satisfy $d(Q_i^{(k)}, Q_i^{(k+1)}) \leqq \alpha_{k+1}$ for any $i \leqq k$. Thus $Q_i = \lim_k Q_i^{(k)}$ exists.

Let $P_1 \subset P_2 \subset P_3 \subset \cdots$ be a sequence of finite partitions with $\bigvee_{i=1}^{\infty} P_i = \Omega$. Let also be given a sequence $(G_i)_{i \geqq 1}$ of (n_i, M_i, δ_i)-uniform towers with ceiling B_i satisfying the conclusion of Lemma 3. (Recall that n_i tends to infinity and δ_i to zero.) One will assume in the sequel that $n_i/M_i \leqq \delta_i/2$ (this is always possible; see Theorem 1).

Step 1. Applying Lemma 4 to the partition P_1 and α_1, in the given sequence $(G_i)_{i \geqq 1}$, one can find a tower G_{i_1} that is $(n_{i_1}, M_{i_1}, \delta_{i_1})$-uniform (one can always assume that $i_1 = 1$ and make this assumption in the sequel) so that $\delta_1 \leqq \alpha_1/3$ and there exists a set $D_1 \subset T^{-n_1}B_1$ with $\lambda(D_1) \geqq (1-\alpha_1)\lambda(B_1)$ and for x in D_1, (3) of Lemma 4 is true for $P = P_1$ and $\alpha = \alpha_1$.

The P_1-name of a point x in the base $T^{-n_1}B_1$ along the tower is $(i_0, i_1, \ldots, i_{n_1})$ if, for $k \leqq n_1$, $T^k x \in p_{i_k}$.

A base A for a pure column with respect to P_1, in the tower G_1, will be a subset of $T^{-n_1}B_1$ of points that have the same P_1-name, that is:

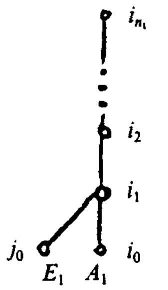
$$A = T^{-n_1}B_1 \bigcap_{j=0}^{j=n_1} T^{-j}p_{i_j} \qquad \text{where } p_{i_j} \text{ is any atom of } P_1. \tag{6}$$

The P_1-name in the column defined by (6) is by definition $(i_0, i_1, \ldots, i_{n_1})$.

A column is said to be *good* if its base contains points in D_1. Note that if a column is good, all points in its base belong to D_1 since they all have the same P_1-name. Let A_1 be the base of any bad column ($A_1 \cap D_1 = \varnothing$). On such a column, P_1 will be replaced by $Q_1^{(1)}$ so that the new $Q_1^{(1)}$-name of points in A_1 will be good in the sense that all the points x in A_1 will satisfy (3) of Lemma 4 for $P = Q_1^{(1)}$ and $\alpha = 2\alpha_1$. In the invertible case, this is easy to do because all the columns are disjoint and one can change names on columns as one wishes. In the non-invertible case, this is not so and one has to do the following:

First, let us fix some order on the bases of the good columns: $E_1, E_2, \ldots, E_l$.

Suppose now initially that $A_1 \cap T^{-1}(TE_1) \neq \varnothing$. (Hence we use the fact that we may suppose that E_1 (or any one of the set used) is irreducible.)

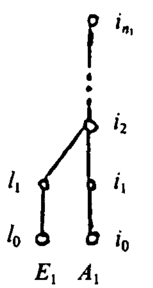


Let $(i_0, i_1, i_2, \ldots, i_{n_1})$ be the P_1-name in A_1.
Let $(j_0, i_1, i_2, \ldots, i_{n_1})$ be the P_1-name in E_1.

We will put in the j_0-th atom of $Q_1^{(1)}$, $A_1 \cap T^{-1}(TE_1)$. We consider then the remaining part of A_1 $(A_1 \setminus T^{-1}(TE_1))$, see if it intersects $T^{-1}(TE_2)$, and put the intersection $(A_1 \setminus T^{-1}(TE_1)) \cap T^{-1}(TE_2)$ in the k_0-th atom of $Q_1^{(1)}$ if the P_1-name in E_2 is $(k_0, i_1, i_2, \ldots, i_{n_1})$. We do the same in the remaining parts of A_1 successively for all the good columns. Suppose that we have not yet exhausted all of A_1, that is,

$$A_1^{(1)} = A_1 \setminus \left(\bigcup_{i=1}^{i=l} A_1 \cap (T^{-1}(TE_i)) \right) \neq \varnothing .$$

We now look at the second level to see if $A_1^{(1)} \setminus T^{-2}(T^2E_1) \neq \varnothing$. (Here again we are using the fact that we may suppose that all the direct images of the set E_1 (or any set we used) are measurable.)



Let $(i_0, i_1, i_2, \ldots, i_{n_{11}})$ be the P_1-name in A_1.
Let $(l_0, l_1, i_2, \ldots, i_{n_1})$ be the P_1-name in E_1.

We put $A_1^{(1)} \cap T^{-2}(T^2E_1)$ in the l_0-th atom of $Q_1^{(1)}$ and $T[A_1^{(1)} \cap T^{-2}(T^2E_1)]$ in the l_1-th atom of $Q_1^{(1)}$. We do this for all the good columns, in order, and then look at the remaining part of A_1, namely $A_1^{(2)}$. Then consider "level 3": if $A_1^{(2)} \cap T^{-3}(T^3E_1) \neq \varnothing$ one changes the P_1-name in the first three levels of $A_1^{(2)}$, according to the P_1-name in E_1. We do this, in order, for all the good columns. In doing so, names of points whose name in the tower was previously changed are not changed again; this comes easily from the definition of the changes. We continue this process up to the upper level of the tower with base A_1. If there remains a part of A_1, one chooses any good P_1-name of length $n_1 + 1$ and defines $Q_1^{(1)}$ accordingly in this remaining part of A_1. (This was the only thing to do in the invertible case.) We successively do the same work in some fixed order for all the different bad columns. At the end, one obtains $Q_1^{(1)}$.

Let us show that $d(Q_1^{(1)}, P_1) \leqq \alpha_1$. To prove it, we will show that on the $(j+1)$-th level (on $T^{-n_1+j}B_1$), the part L_j where the P_1-name differs from the $Q_1^{(1)}$-name satisfies

$$T^{-j}L_j \subset T^{-n_1}B_1 - D_1 \tag{7}$$

so that $\lambda(L_j) = \lambda(T^{-j}L_j) \leqq \alpha_1\lambda(B_1)$ and $\lambda(\bigcup_{j=0}^{j=n_1} L_j) \leqq \alpha_1$. A point in L_j is in a bad column C_1. If (7) is not true, there is a part L_j' of L_j so that $T^{-j}L_j' \subset E_0$ and $L_j' \subset T^jA_1$, for some base E_0 of a good column and base A_1 of a bad column. It results that

$$A_1 \cap (T^{-j}(T^jE_0)) \neq \varnothing$$

so that, by our construction, we should have changed P_1-names into different $Q_1^{(1)}$-names up to the j-th level (on the $(j+1)$-th level names should be identical). This contradicts the construction and proves (7). Because the tower G_1 was (n_1, M_1, δ_1)-uniform and every $Q_1^{(1)}$-name in the tower is now "good up to $2\alpha_1$" for the ergodic theorem, for the atoms of $Q_1^{(1)}$, one easily obtains that $Q_1^{(1)}$ is $(1, M_1, 3\alpha_1)$-uniform. (In a $Q_1^{(1)}$-name of length M_1, except $\delta_1 M_1$ of the places, one is in a G_1-tower and not considering, if necessary, the part in the first n_1 or last n_1 places, one can suppose that the G_1-towers appearing in this $Q_1^{(1)}$-name are complete. From $n_1/M_1 \leqq \delta_1/6$ and $\delta_1 \leqq \alpha_1/3$, one gets the desired result. The fact that the measure of the atoms of $Q_1^{(1)}$ is slightly different from that of the atoms of P_1 is the reason for the fact that names are good only "up to $2\alpha_1$".)

Step 2. We replace P_2 by $P_2 \vee Q_1^{(1)}$ to have $P_2 \supset Q_1^{(1)}$ and fix α_2. We will change P_2 into $Q_2^{(2)}$ so that $d(P_2, Q_2^{(2)}) \leqq \alpha_2$. Because $Q_1^{(1)} \subset P_2$, to every atom of P_2 corresponds an atom of $Q_1^{(1)}$ and every atom of $Q_1^{(1)}$ is a union of atoms of P_2. Using this correspondence $Q_2^{(2)}$ will define $Q_1^{(2)}$ such that $Q_1^{(2)} \subset Q_2^{(2)}$. For instance, if the first atom of $Q_1^{(1)}$ was the union of the second and fourth atoms of P_2, the first atom of $Q_1^{(2)}$ will be the union of the second and fourth atoms of $Q_2^{(2)}$. Our goal is thus to define $Q_2^{(2)}$.

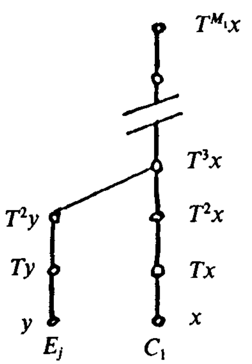
Let us apply Lemma 4 for the partition $P_2 \vee T^{-1}P_2$ and α_2 and find, as above, a tower that we may suppose to be G_2 that is (n_2, M_2, δ_2)-uniform. If $T^{-n_2}B_2$ is the base of the tower, there is a set $D_2 \subset T^{-n_2}B_2$ with $\lambda(D_2) \geqq (1 - \alpha_2)\lambda(B_2)$ and, for x in D_2, (3) of Lemma 4 is true for $P_2 \vee T^{-1}P_2$ and α_2. A base of a column for $P_2 \vee T^{-1}P_2 \vee B_1'$ in the tower will be a subset of $T^{-n_2}B_2$ of points that have the same $P_2 \vee T^{-1}P_2 \vee B_1'$-name in the tower where B_1' is the partition (B_1, B_1^c).

One first deletes from $T^{-n_1}B_1$ the part of it that intersects $\bigcup_{j=0}^{j=n_1} T^{-j}B_2$ (the first n_1 levels of G_2) and $\bigcup_{j=-n_2-n_1}^{j=-n_2-1} T^{-j}B_2$ (there does not remain any tower like G_1 in the last n_1 levels of G_2). At that time there are no partial G_1-towers in G_2.

On a column containing only points in D_2 (good columns), no change is made.

On bad columns with base A_1 ($A_1 \cap D_2 \neq \emptyset$), changes are done as in Step 1: Consider the first base of a good column E_j so that $A_1 \cap T^{-1}(TE_j) \neq \emptyset$ (as before, some order is fixed on the base of good columns). If $(i_0, i_1, \ldots, i_{n_2-1}, i_{n_2})$ is the $P_2 \vee T^{-1}P_2 \vee B_1'$-name in A_1 and $(j_0, i_1, \ldots, i_{n_2-1}, i_{n_2})$ is the $P_2 \vee T^{-1}P_2 \vee B_1'$-name in E_j, we put $A_1 \cap T^{-1}(TE_j)$ in the j_0-th atom of $Q_2^{(2)}$. This process is then continued, as in Step 1. In doing so, T_1 (and B_1) will be changed into $T_1^{(2)}$ (and $B_1^{(2)}$). For instance, if at some stage in this process, for a given k with $H = A_1 \cap T^{-k}(T^kE_j)$, $H \neq \emptyset$, A_1 the given base of a bad column and E_j the base of some good column, we will erase from B_1 all the points in T^lH for $l < k$ such that $T^lH \subset B_1$ and put into $B_1^{(2)}$ all the points of T^mH, for $m < k$ such that $T^mE_j \subset B_1$. As in Step 1, it is clear that the partition $Q_2^{(2)}$, so obtained, satisfies $d(Q_2^{(2)}, P_2) \leqq \alpha_2$ and is $(2, M_2, 3\alpha_2)$-uniform. The fact that uniformity from Step 1 is kept for $Q_1^{(2)}$ is proved as follows.

It is first worth noting the following point: When a point x and its M_1 first images under T are in a given bad column C_1 (from Step 2), its M_1-name for $Q_1^{(2)}$ is in fact identical to the M_1-name for $Q_1^{(2)}$ for some point y in a good column E_j. This comes easily from the definition of the changes (see the diagram below), so the "properties of uniformity" of x are not changed in Step 2.



The only real change is thus when $x, Tx, \dots, T^{M_1}x$ intersects the boundary of F_2: $B_2 \cup T^{-n_2}B_2$ (the following argument is similar to the invertible case; see [Rs2] or [W]). These M_1 successive images of x are intersecting at most two F_2-towers, so that in the worst case these M_1 images can be divided into three parts: the first is the part in the first F_2-tower, the second is between the two F_2-towers, and the third is in the second F_2-tower. In the first part there are at most $\delta_1 M_1 + n_1$ places not in an F_1-tower. $\delta_1 M_1$ comes from the fact that F_1 is (n_1, M_1, δ_1)-uniform, n_1 from the F_1-tower that may have been erased in the first n_1 places of the F_2-tower. Similarly, in the third part, there are at most $\delta_1 M_1 + n_1$ places not in an F_1-tower and in the second part there are at most $\delta_1 M_1 + 2n_1$ places not in an F_1-tower (in this part, one is twice near the boundary of some F_2-tower). Using then $2n_1/M_1 \leqq \delta_1 \leqq \alpha_1/3$, it becomes clear that $Q_1^{(2)}$ remains $(1, M_1, 5\alpha_1)$-uniform.

Step k. Changing P_k into $P_k \vee Q_k^{(k-1)}$, the following holds:

$$Q_1^{(k)} \subset Q_2^{(k)} \subset \cdots \subset Q_{k-1}^{(k)} \subset P_k. \tag{8}$$

Let α_k be fixed; doing the same work as in Step 2 with P_k and $\bigvee_{j=0}^{j=k-1} T^{-j}P_k$ one obtains $Q_k^{(k)}$, and the same way as in Step 2, from (8), one finds a sequence

$$Q_1^{(k-1)} \subset Q_2^{(k-1)} \subset \cdots \subset Q_{k-1}^{(k-1)} \subset P_k$$

satisfying: For $j \leqq k$, $d(Q_j^{(k-1)}, Q_j^{(k)}) \leqq \alpha_k$, $d(P_k, Q_k^{(k)}) \leqq \alpha_k$, $Q_j^{(k)}$ is $(l, M_j, 5\alpha_j)$-uniform for $j \leqq l \leqq k$ and, finally, $Q_k^{(k)}$ is (k, M_k, α_k)-uniform. The fact that the uniform properties from prior steps are retained comes, as in the invertible case (see again [Rs2] or [W]), from the fact, obtained in Lemma 3, that for any j and any x in the M_j first images of x under T, there are at most two boundaries $(B_l \cup T^{-n_l}B_l)$ of a tower T_l, for one given l. Letting $Q_j = \lim_k Q_j^{(k)}$, the Q_j are easily seen to be uniform and $\bigvee_j Q_j$ generate Ω. (For a more detailed proof see theorem 11 and corollary 12 of [Rs2].) This ends the proof of Theorem 1.

From Theorem 1, it is an easy matter (see [H–R] or [Rs2]) to obtain our final result:

THEOREM 3. *Let T be an ergodic measure preserving transformation on (X, Ω, λ), T not necessarily invertible. There exists a strictly ergodic S acting on (Y, Θ, ν) where Y is compact such that (X, Ω, λ, T) is measure isomorphic to (Y, Θ, ν, S).*

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